

## AN IMPRIMITIVITY THEOREM FOR PARTIAL ACTIONS

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ABSTRACT. We define proper, free and commuting partial actions on upper semicontinuous bundles of  $C^*$ -algebras. With such, we construct the  $C^*$ -algebra induced by a partial action and a partial actions on that algebra. Using those action we give a generalization, to partial actions, of Raeburn's Symmetric Imprimitivity Theorem [10].

## INTRODUCTION

The main idea of this article appear in the following example. Let  $\beta$  be a continuous, free and proper action of a locally compact and Hausdorff (LCH) group  $G$  on a LCH space  $Y$ . This gives us a continuous action of  $G$  on the continuous functions vanishing at infinity of  $Y$ ,  $C_0(Y)$ . If  $Y/G$  is the orbit space of  $Y$ , then Green's Theorem [11] implies  $C_0(Y/G)$  is strongly Morita equivalent to the crossed product  $C_0(Y) \rtimes_{\beta} G$ .

Now consider an open subset  $X \subset Y$  such that  $\cup\{\beta_t(X) : t \in G\} = Y$ . Lets call  $\alpha$  the restriction of  $\beta$  to  $X$ . That is, for every  $t \in G$  set  $\alpha_t : X \cap \beta_{t^{-1}}(X) \rightarrow X \cap \beta_t(X)$ ,  $x \mapsto \beta_t(x)$ . This is an example of a partial action. Now consider the open set  $\Gamma := \{(t, x) \in G \times X \mid \beta_{t^{-1}}(x) \in X\} \subset G \times Y$ . The crossed product  $C_0(X) \rtimes_{\alpha} G$  is the closure of  $C_c(\Gamma) \subset C_c(G, Y)$  in  $C_0(Y) \rtimes_{\beta} G$ . It is strongly Morita equivalent to  $C_0(Y) \rtimes_{\beta} G$  [2, 3].

Putting all together, we conclude that  $C_0(X) \rtimes_{\alpha} G$  is strongly Morita equivalent to  $C_0(Y/G)$ . The objective of the present work is to generalize the previous idea to the case where we just know  $X$ ,  $G$  and  $\alpha$ . That is,  $\alpha$  is a partial action of  $G$  on  $X$ .

The outline of this work is as follows. In Section 1 we give the definitions of free, proper and commuting partial actions and prove some basic results involving those concepts, it is based on [1, 2]. In the second section we define partial actions on upper semicontinuous  $C^*$ -bundles and, with such, construct the induced  $C^*$ -algebra of a partial action and partial actions on those induced algebras. Here we follow Raeburn's work [10]. Finally, we prove our main theorem which is a generalization, to partial actions, of Raeburn's Theorem [10]. On a first read, to understand the basic ideas, we suggest the reader to consider bundles of the form  $X \times \mathbb{C}$  ( $X$  is a topological space and  $\mathbb{C}$  the complex numbers) with trivial action on  $\mathbb{C}$ .

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## 1. PROPERTIES OF PARTIAL ACTIONS

Through this work the letters  $G$ ,  $H$  and  $K$  will denote LCH topological groups and  $X$ ,  $Y$  topological spaces. When any additional topological property is required it will be explicitly mentioned (this will never happen for the groups).

This section is a brief resume of some results contained in [2] and in the PHD Thesis [1], for that reason some proof will be omitted. We start by recalling the definition of partial action.

**Definition 1.1** ([5, 2, 1]). A pair  $\alpha = (\{X_t\}_{t \in H}, \{\alpha_t\}_{t \in H})$  is a partial action of  $H$  on  $X$  if, for every  $t, s \in H$ :

- (1)  $X_t$  is a subset of  $X$  and  $X_e = X$  ( $e$  being the identity of  $H$ ).
- (2)  $\alpha_t : X_t \rightarrow X_{t^{-1}}$  is a bijection and  $\alpha_e = \text{id}_X$  (the identity on  $X$ ).
- (3) If  $x \in X_{t^{-1}}$  and  $\alpha_t(x) \in X_{s^{-1}}$ , then  $x \in X_{(st)^{-1}}$  and  $\alpha_{st}(x) = \alpha_s \circ \alpha_t(x)$ .

The domain of  $\alpha$  is the set  $\Gamma_\alpha := \{(t, x) \in H \times X \mid x \in X_{t^{-1}}\}$ . Recall  $\alpha$  is continuous if  $\Gamma_\alpha$  is open in  $H \times X$  and the function, also called  $\alpha$ ,  $\Gamma_\alpha \rightarrow X$ ,  $(t, x) \mapsto \alpha_t(x)$ , is continuous. The graph of the partial action  $\alpha$ ,  $\text{Gr}(\alpha)$ , is the graph of the function  $\alpha : \Gamma_\alpha \rightarrow X$ . We say  $\alpha$  has closed graph if  $\text{Gr}(\alpha)$  is closed in  $H \times X \times X$ .

Take two continuous partial actions of  $H$ ,  $\alpha$  and  $\beta$ , on the spaces  $X$  and  $Y$  respectively. A morphism  $f : \alpha \rightarrow \beta$  is a continuous function  $f : X \rightarrow Y$  such that for every  $t \in H$  :  $f(X_t) \subset Y_t$  and the restriction of  $\beta_t \circ f$  to  $X_{t^{-1}}$  equals  $f \circ \alpha_t$ .

Given  $\beta$  as before and a non empty open set  $Z \subset Y$ , the restriction of  $\beta$  to  $Z$  is the continuous partial action of  $H$  on  $Z$  given by  $\gamma_t : Z \cap \beta_{t^{-1}}(Z) \rightarrow Z \cap \beta_t(Z)$ ,  $z \mapsto \beta_t(z)$ .

Up to isomorphism of partial actions, every continuous partial action can be obtained as a restriction of a global action. That is, given  $\alpha$  as before there exists a global and continuous action of  $H$  on a topological space  $Y$ ,  $\beta$ , and an open set  $Z \subset Y$  such that  $\alpha$  is isomorphic to the restriction of  $\beta$  to  $Z$ . If in addition  $Y = \cup\{\beta_t(Z) \mid t \in H\}$ , we say  $\beta$  is an *enveloping action* of  $\alpha$ . Enveloping actions exist and are unique up to isomorphism of (partial) actions [2, 1]. The enveloping<sup>1</sup> action of  $\alpha$  is denoted  $\alpha^e$  and the space where it acts  $X^e$ , we also think  $X$  is an open set of  $X^e$  and  $\alpha$  is the restriction of  $\alpha^e$  to  $X$ .

The orbit of a subset  $U \subset X$  by  $\alpha$  is the set  $HU := \cup\{\alpha_t(U \cap X_{t^{-1}}) \mid t \in G\}$ . The orbit of a point  $x \in X$  is the orbit of the set  $\{x\}$  and is denoted  $Hx$ . If we want to emphasize the name of the action we write  $\alpha Hx$ . The orbits of two points are equal or disjoint and the union of all of them is equal to  $X$ . With this partition of  $X$  we construct the quotient space  $X/H$  with the quotient topology, this is the *orbit space* of  $\alpha$ . The canonical projection  $X \rightarrow X/H$  is continuous, surjective and open. The function  $X/H \rightarrow X^e/H$ ,  $\alpha Hx \rightarrow \alpha^e Hx$  is a homeomorphism.

Raeburn's Symmetric Imprimitivity Theorem involves free, proper and commuting actions. We now give the corresponding definitions for partial actions. We refer the reader to [1] to a more detailed exposition of these concepts.

The *stabilizer* of a point  $x \in X$  is the set  $H_x := \{t \in H \mid x \in X_{t^{-1}}, \alpha_t(x) = x\}$ . It is easy to see that  $H_x$  is a subgroup of  $H$ , not necessarily closed if the action is not global. A partial action is *free* if the stabilizer of every point is the set  $\{e\}$ . A partial action is free if and only if its enveloping action is free.

<sup>1</sup>We say "the" enveloping action because it is unique up to isomorphisms.

The next concept we define is commutativity. We will have two continuous partial actions,  $\alpha$  and  $\beta$ , of  $H$  and  $K$ , on  $X$ . As we do not want any confusions, we will use the notation  $\alpha_s : X_{s^{-1}}^H \rightarrow X_s^H$  and  $\beta_t : X_{t^{-1}}^K \rightarrow X_t^K$ , for  $s \in H$  and  $t \in K$ .

We say  $\alpha$  and  $\beta$  *commute* if for every  $(s, t) \in H \times K$  (i)  $\alpha_s(X_{s^{-1}}^H \cap X_t^K) = \beta_t(X_{t^{-1}}^K \cap X_s^H)$  and (ii)  $\alpha_s \circ \beta_t(x) = \beta_t \circ \alpha_s(x)$ , for every  $x \in \alpha_{s^{-1}}(X_s^H \cap X_{t^{-1}}^K)$ . This definition expresses the fact that we can compute  $\alpha_s \beta_t(x)$  if and only if we can compute  $\beta_t \alpha_s(x)$ , and in that case  $\alpha_s \beta_t(x) = \beta_t \alpha_s(x)$ . As we can see, if both actions are global, this is the usual notion of commuting actions.

Recall [6] a subset  $U \subset X$  is said  $\alpha$ -invariant if  $\alpha_t(X_{t^{-1}}^H \cap U) \subset U$ , for every  $t \in H$ . Condition (i) of the previous definition implies  $X_s^K$  is  $\alpha$ -invariant for every  $s \in K$ .

An important property of commuting global actions is that we can define an action of the product group, this is also true for partial actions.

**Lemma 1.2** (cf. [1] Proposição 4.35). *If  $\alpha$  and  $\beta$  commute then there is a continuous partial action,  $\alpha \times \beta$ , of  $G := H \times K$  on  $X$  such that, for every  $(s, t) \in G$ ,*

- (1)  $X_{(s,t)} = \beta_t(X_{t^{-1}}^K \cap X_s^H) = \alpha_s(X_{s^{-1}}^H \cap X_t^K)$  and
- (2)  $\alpha \times \beta_{(s,t)} = \alpha_s \circ \beta_t$ .

*Proof.* The fact that  $\mu := \alpha \times \beta$  is a partial action (not necessarily continuous) is an easy consequence of the fact that  $\alpha$  and  $\beta$  are commuting partial actions ([1] Proposição 4.35). We just have to deal with the continuity.

To show  $\Gamma_\mu$  is open in  $G \times X$  notice that the set  $\Gamma_\beta^{-1} := \{(t, x) \in K \times X \mid x \in X_t^K\}$  is open in  $K \times X$ , and define:

$$\begin{aligned} \pi_H : H \times K \times X &\rightarrow H \times X, (h, k, x) \mapsto (h, x), \\ \pi_K : H \times K \times X &\rightarrow K \times X, (h, k, x) \mapsto (k, x), \text{ and} \\ F : \pi_K^{-1}(\Gamma_\beta) &\rightarrow \pi_K^{-1}(\Gamma_\beta^{-1}), (h, k, x) \mapsto (h, k, \beta_k(x)). \end{aligned}$$

It is easy to see that the three functions are continuous. So, the domain and range of  $F$  are open in  $H \times K \times X$  and  $\Gamma_\mu = F^{-1}(\pi_K^{-1}(\Gamma_\beta^{-1}) \cap \pi_H^{-1}(\Gamma_\alpha))$  is open in  $G \times X$ . Finally, the continuity of  $\mu : \Gamma_\mu \rightarrow X$  follows from that of  $\alpha : \Gamma_\alpha \rightarrow X$  and  $\beta : \Gamma_\beta \rightarrow X$ .  $\square$

Here is another property of partial actions we will use.

**Lemma 1.3.** *If  $\alpha$  and  $\beta$  commute then there is a partial action of  $H$  on  $X/K$ , called  $\hat{\alpha}$ , such that for every  $s \in H$*

- (1)  $(X/K)_s := KX_s^H$  and
- (2)  $\hat{\alpha}_s(Kx) = K\alpha_s(x)$  for any  $x \in X_{s^{-1}}^H$ .

*Proof.* The first step is to show we can define  $\hat{\alpha}$  as in (1) and (2). Define the function  $F : \Gamma_\alpha \rightarrow H \times X/K$  as  $F(s, x) = (s, Kx)$ . This map is open and continuous. The domain of  $\hat{\alpha}$  will be the image of  $F$ ,  $\Gamma_{\hat{\alpha}} := \text{Im}(F)$ , which is an open set.

Now define  $S : \Gamma_\alpha \rightarrow X/K$  in such a way that  $(s, x) \mapsto K\alpha_s(x)$ , and consider (on  $\Gamma_\alpha$ ) the equivalence relation  $u \sim v$  if  $F(u) = F(v)$ . The function  $S$  is constant in the classes of  $\sim$ , and the quotient space  $\Gamma_\alpha / \sim$  is homeomorphic to  $\Gamma_{\hat{\alpha}}$  through the map defined by  $F$ . So, there is a unique continuous map  $\Gamma_{\hat{\alpha}} \rightarrow X/K$  such that  $(s, Kx) \mapsto K\alpha_s(x)$ . This is the partial action  $\hat{\alpha}$  we are looking for.

It remains to be shown that  $\hat{\alpha}$  is a partial action. Properties (1) and (2) of Definition 1.1 are easy to prove, for (3) recall every  $X_s^H$  is  $\beta$ -invariant.  $\square$

Assume for a moment we have a continuous global action of  $H$  on  $X$ . It is immediate that its domain, being equal to  $H \times X$ , is a closed and open (clopen) set of  $H \times X$ . That is not always the case for partial actions.

**Definition 1.4.** A partial action,  $\alpha$ , of  $H$  on  $X$  has *closed domain* if  $\Gamma_\alpha$  is closed in  $H \times X$ .

**Lemma 1.5.** *If  $X$  is Hausdorff and  $\alpha$  is continuous, the following conditions are equivalent:*

- (1)  $\alpha$  has closed domain.
- (2) The enveloping space  $X^e$  is Hausdorff and  $X$  is closed in  $X^e$ .

*Proof.* We start by proving (1) $\Rightarrow$ (2). Recall [2]  $X^e$  is Hausdorff if  $\alpha$  has closed graph. Consider the function  $F : H \times X \times X \rightarrow H \times X$ ,  $(s, x, y) \mapsto (s, x)$ . The set  $F^{-1}(\Gamma_\alpha)$  is closed in  $H \times X \times X$ . Now,  $\text{Gr}(\alpha)$  is closed in  $F^{-1}(\Gamma_\alpha)$  because it is the pre-image of the diagonal  $\{(x, x) \mid x \in X\} \subset X \times X$  by the continuous function  $F^{-1}(\Gamma_\alpha) \rightarrow X \times X$ ,  $(t, x, y) \mapsto (\alpha_t(x), y)$ . This implies  $\alpha$  has closed graph.

To show  $X$  is closed in  $X^e$  take a net contained in  $X$ ,  $\{x_i\}_{i \in I}$ , converging to a point  $x \in X^e$ . There exists  $t \in H$  such that  $\alpha_t^e(x) \in X$ . By the continuity of  $\alpha^e$  there is an  $i_0$  such that  $(t^{-1}, \alpha_t^e(x_i)) \in \Gamma_\alpha$  for  $i \geq i_0$ . Then  $(t^{-1}, \alpha_t^e(x))$ , being the limit of  $\{(t^{-1}, \alpha_t^e(x_i))\}_{i \geq i_0}$ , belongs to  $\Gamma_\alpha$ . Finally  $x = \alpha_{t^{-1}}\alpha_t^e(x) \in X$ .

For the converse notice three facts: the topology of  $H \times X$  is the topology relative to  $H \times X^e$ ,  $(\alpha^e)^{-1}(X)$  is closed in  $H \times X^e$  and  $\Gamma_\alpha = H \times X \cap (\alpha^e)^{-1}(X)$ . So we clearly have that  $\Gamma_\alpha$  is closed in  $H \times X$ .  $\square$

The previous lemma characterizes the continuous partial actions with closed domain on Hausdorff spaces, as those arising as the restriction of a global action on a Hausdorff space to a clopen set.

**Lemma 1.6.** *Given two continuous and commuting partial actions, both with closed domain, the partial action of the product group (as defined on Lemma 1.2) has closed domain.*

*Proof.* In the proof of Lemma 1.2 we showed  $\Gamma_{\alpha \times \beta}$  is open, use the same arguments changing the word “open” for “closed”.  $\square$

A dynamical system (DS for short) is a tern  $(Y, H, \beta)$  where  $\beta$  is a continuous action of  $H$  on  $Y$ , where  $H$  and  $Y$  are LCH. The natural extension to partial actions is the following one.

**Definition 1.7.** The tern  $(X, H, \alpha)$  is a *partial dynamical system* (PDS) if  $\alpha$  is a continuous partial action of  $H$  on  $X$  and both  $(H$  and  $X)$  are LCH.

**Lemma 1.8.** *If  $(X, H, \alpha)$  is a PDS then  $(X^e, H, \alpha^e)$  is a DS if and only if  $\alpha$  has closed graph.*

*Proof.* By Theorem 1.1. of [2] every point of  $X^e$  has a neighbourhood homeomorphic to  $X$ . So, every point of  $X^e$  has a local basis of compact neighbourhoods. As  $\alpha$  is continuous and  $H$  is LCH,  $(X^e, H, \alpha^e)$  is a DS if and only if  $X^e$  is Hausdorff. By Proposition 1.2. of [2]  $X^e$  is Hausdorff if and only if  $\alpha$  has closed graph.  $\square$

A PDS  $(X, H, \alpha)$  is *proper* if the function  $F_\alpha : \Gamma_\alpha \rightarrow X$ ,  $(t, x) \mapsto (x, \alpha_t(x))$ , is proper (the pre-image of a compact set is compact). This definition, and part of the next Lemma, are taken from [1].

**Lemma 1.9.** *Given a PDS  $(X, H, \alpha)$ , the following statements are equivalent:*

- (1) *The system is proper.*
- (2) *Every net contained in  $\Gamma_\alpha$ ,  $\{(t_i, x_i)\}_{i \in I}$ , such that  $\{(x_i, \alpha_{t_i}(x_i))\}_{i \in I}$  converges to some point of  $X \times X$ , has a subnet converging to a point of  $\Gamma_\alpha$ .*
- (3)  *$\alpha$  has closed graph and the enveloping DS  $(X^e, H, \alpha^e)$  is proper.*

*Proof.* The equivalence between (1) and (3) is proved in [1] (Proposição 4.62). The equivalence between (1) and (2) is proved as in Lemma 3.42 of [13].  $\square$

It is a known fact that the orbit space of a proper DS is a LCH space, this is also true for PDS.

**Lemma 1.10.** *If  $(X, H, \alpha)$  is a proper PDS then  $X/H$  is a LCH space.*

*Proof.* By the previous Lemma  $(X^e, H, \alpha^e)$  is a proper DS. So,  $X^e/H$  is LCH. But  $X/H$  is homeomorphic to  $X^e/H$  and so is LCH.  $\square$

The next result follows immediately from the previous ones.

**Lemma 1.11.** *Let  $(X, H, \alpha)$  and  $(X, K, \beta)$  be commuting PDS (that is,  $\alpha$  and  $\beta$  commute). If  $\beta$  is proper then  $(X/K, H, \hat{\alpha})$  is a PDS, where  $\hat{\alpha}$  is the partial action defined on Lemma 1.3.*

## 2. PARTIAL ACTIONS ON BUNDLES OF $C^*$ -ALGEBRAS.

The definition of upper semicontinuous  $C^*$ -bundle we are going to use is Definition C.16 of [13] (notice that we do not require the base space to be Hausdorff). From now on  $\mathbf{B} = \{B_x\}_{x \in X}$  and  $\mathbf{C} = \{C_y\}_{y \in Y}$  will be upper semicontinuous  $C^*$ -bundles. The projections of  $\mathbf{B}$  and  $\mathbf{C}$  will be denoted  $p : B \rightarrow X$  and  $q : C \rightarrow Y$ , respectively.

The set of continuous and bounded sections of the bundle  $\mathbf{B}$  will be denoted  $C_b(\mathbf{B})$  (this notation differs from that of [13]). Similarly,  $C_0(\mathbf{B})$  is the set of continuous sections vanishing at infinity (C.21 [13]) and  $C_c(\mathbf{B})$  the set of continuous sections of compact support. When  $X$  is a LCH space,  $C_b(\mathbf{B})$  and  $C_0(\mathbf{B})$  are  $C^*$ -algebras with the supremum norm and  $C_c(\mathbf{B})$  is a dense  $*$ -sub algebra of  $C_0(\mathbf{B})$ .

**Definition 2.1.** A partial action of  $H$  on  $\mathbf{B}$  is a pair  $(\alpha, \cdot)$ , where  $\alpha$  and  $\cdot$  are continuous partial actions of  $H$  on  $B$  and  $X$ , respectively, satisfying

- (1)  $p^{-1}(X_t) = {}_tB$  for every  $t \in H$ . Here  ${}_tB$  is the range of  $\alpha_t$ .
- (2)  $p$  is a morphism of partial actions (Definition 1.1 of [2]).
- (3) The restriction of  $\alpha_t$  to a fiber is a morphism of  $C^*$ -algebras, for each  $t \in H$ .

Notice that  $\cdot$  is determined by  $\alpha$ . For that reason, with abuse of notation, we name  $\alpha$  the pair  $(\alpha, \cdot)$ . We say  $\alpha$  is global if the partial action on the total space is a global action or, what is the same, if  $\cdot$  is global.

The domain of the partial action on the total and base space will be denoted  $\Gamma(B, \alpha)$  and  $\Gamma(X, \alpha)$ , respectively.

**Example 2.2.** Let  $(X, H, \cdot)$  be a PDS and  $(A, H, \gamma)$  a  $C^*$ -DS, that is,  $A$  is a  $C^*$ -algebra and  $\gamma : H \rightarrow \text{Aut}(A)$  is a strongly continuous action. With such define the trivial bundle  $p : A \times X \rightarrow X$ , where  $p(a, x) = x$ . All the fibers of this bundle, called  $\mathbf{B}$ , are isomorphic to  $A$  by the maps  $A \rightarrow B_x$ ,  $a \mapsto (a, x)$ . We define a global action of  $H$  on  $\mathbf{B}$  by setting  $\alpha_t : A \times X_{t^{-1}} \rightarrow A \times X_t$ ,  $(a, x) \mapsto (\gamma_t(a), t \cdot x)$ .

I would like to emphasize that, from now on, we are going to use the letters  $\alpha$  and  $\beta$  for actions on total spaces. The actions on the base spaces will be denoted  $\cdot$  and  $\star$ . We will write  $\alpha_t(a)$  and  $t \cdot x$ , similarly with  $\beta$  and  $\star$ .

If  $\beta = (\beta, \star)$  is a partial action of  $H$  on  $\mathbf{C}$ , a morphism  $(F, f) : \alpha \rightarrow \beta$  is a pair of continuous functions,  $F : B \rightarrow C$  and  $f : X \rightarrow Y$ , such that: both are morphism of partial actions,  $q \circ F = f \circ p$  and the restriction of  $F$  to each fiber is a morphism of  $C^*$ -algebras. Naturally, the composition of morphisms is the composition of functions (on each coordinate).

Following [2] we can define the restriction of actions. Let  $\beta = (\beta, \star)$  be a global action of  $H$  on  $\mathbf{B}$  and  $U$  an open subset of  $X$ . Consider the restriction bundle  $\mathbf{B}_U = \{B_u\}_{u \in U}$  with the partial action  $\beta_U$ , which is the pair formed by the restriction of the actions of  $H$  to  $p^{-1}(U)$  and  $U$ . Notice we have obtained a partial action because  $p^{-1}(U) \cap \beta_t(p^{-1}(U)) = p^{-1}(U \cap t \star U)$ , for every  $t \in H$ .

Rephrasing Theorem 1.1 of [2] we get

**Theorem 2.3.** *For every continuous partial action  $\alpha$  of  $H$  on an upper semicontinuous  $C^*$ -bundle  $\mathbf{B}$ , there exists a tern  $(\iota_X, \iota_B, \alpha^e)$  such that  $\alpha^e$  is an action of  $H$  on an upper semicontinuous  $C^*$ -bundle  $\mathbf{B}^e$ , and  $(\iota_X, \iota_B) : \alpha \rightarrow \alpha^e$  is a morphism, such that for any morphism  $\psi : \alpha \rightarrow \beta$ , where  $\beta$  is an action of  $H$  (on an upper semicontinuous  $C^*$ -bundle), there exists a unique morphism  $\psi^e : \alpha^e \rightarrow \beta$  such that  $\psi^e \circ (\iota_X, \iota_B) = \psi$ .*

Moreover, the pair  $(\iota_X, \iota_B, \alpha^e)$  is unique up to canonical isomorphisms, and

- (1)  $\iota_X(X)$  is open in  $X^e$ .
- (2)  $(\iota_X, \iota_B) : \alpha \rightarrow (\alpha^e)_{\iota_X(X)}$  is an isomorphism.
- (3)  $X^e$  is the orbit of  $\iota_X(X)$ .
- (4)  $\mathbf{B}^e$  is a continuous  $C^*$ -bundle if and only if  $\mathbf{B}$  is.

*Proof.* Let  $(\iota_X, \cdot^e)$  and  $(\iota_B, \alpha^e)$  be the pairs given by Theorem 1.1 of [2] for  $\cdot$  and  $\alpha$ . We also have a morphism  $p^e : \alpha^e \rightarrow \cdot^e$ . Notice  $p^e$  is surjective because is a morphism and the orbit of  $\iota_X(X)$  equals  $X^e$ . Again, as  $B^e$  is the orbit of  $\iota_B(B)$  and  $p^e$  is a morphism, to prove  $p^e$  is open we only have to see that  $p^e \circ \alpha_t^e \circ \iota_B$  is open, for every  $t \in H$ . But this is true because, if  $U$  is open in  $B$

$$p^e \circ \alpha_t^e \circ \iota_B(U) = t \cdot^e (\iota_X(p(U))),$$

the last being an open set.

We have proved  $B^e$  fibers over  $X^e$ . We now give a structure of  $C^*$ -algebra to each fiber of  $B^e$ . Let  $x$  be an element of  $X^e$ , take  $t \in H$  such that  $t \cdot^e x \in \iota_X(X)$  and define the  $C^*$ -structure on  $B_x^e$  as the unique making  $\alpha_{t^{-1}}^e \circ \iota_B : B_{\iota_X^{-1}(t \cdot^e x)} \rightarrow B_x^e$  an isomorphism of  $C^*$ -algebras. This is independent of the choice of  $t$  because  $\alpha$  acts as isomorphism of  $C^*$ -algebras on the fibers of  $\mathbf{B}$ .

To prove the norm of  $\mathbf{B}^e$  is semicontinuous notice that, given  $\varepsilon > 0$ , the set  $\{b \in B^e : \|b\| < \varepsilon\}$  equals the open set  $\bigcup_{t \in H} \alpha_t^e \circ \iota_B(\{b \in B : \|b\| < \varepsilon\})$ . In fact, a similar argument shows the norm of  $\mathbf{B}^e$  is continuous if and only if the norm of  $\mathbf{B}$  is continuous. This suffices to prove property (4) of the thesis.

We now indicate how to prove the continuity of the product, for the other operations there are analogous proofs. Set  $D^e := \{(a, b) \in B^e \times B^e : p^e(a) = p^e(b)\}$ . We prove the continuity of  $D^e \rightarrow B^e$ ,  $(a, b) \mapsto ab$ , locally. Fix  $(a, b) \in D^e$ , we may assume  $p(a) = t \cdot^e x$  for some  $x \in X$  and  $t \in H$ . The product is continuous on  $(a, b)$  because  $U := (\alpha_t \circ \iota_B(B) \times \alpha_t \circ \iota_B(B)) \cap D^e$  is open in  $D^e$ , and the restriction of

the product to  $U$  is the continuous function

$$(c, d) \mapsto \alpha_t^e \circ \iota_B [(\iota_B)^{-1} \circ \alpha_{t^{-1}}^e(c) + (\iota_B)^{-1} \circ \alpha_{t^{-1}}^e(d)].$$

Up to here we have constructed an upper semicontinuous  $C^*$ -bundle  $\mathbf{B}^e = \{B_x^e\}_{x \in X^e}$ . By the previous construction we also have that  $(\alpha^e, \iota^e)$  is a global action of  $H$  on  $\mathbf{B}^e$ , and  $(\iota_X, \iota_B) : \alpha \rightarrow \alpha^e$  is a morphism. Except for property (2) of the thesis, everything follows immediately from the previous constructions and Theorem 1.1 of [2].

To prove property (2) it suffices to see that  $(p^e)^{-1}(\iota_X(X)) = \iota_B(B)$ . We clearly have the inclusion  $\supset$ , for the other one let  $b \in (p^e)^{-1}(\iota_X(X))$ . We may suppose  $b = \alpha_t^e(\iota_B(c))$  for some  $c \in B$  and  $t \in H$ . As  $p^e(b) = p^e(\alpha_t^e(\iota_B(c))) \in \iota_B(B)$ , we have

$$p^e(b) = p^e(\alpha_t^e(\iota_B(c))) = t \cdot^e \iota_X(p(c)) \in \iota_X(X).$$

So,  $p(c) \in X_{t^{-1}}$ . This implies  $b = \iota_B(\alpha_t(c)) \in \iota_B(B)$ .  $\square$

The non commutative analogue of PDS's are the  $C^*$ -PDS's, they are terns  $(A, G, \gamma)$  formed by a  $C^*$ -algebra  $A$ , a LCH group  $G$  and a partial action  $\gamma$  of  $G$  on  $A$  (Definition 2.2 of [2], for a more general definition see [5]).

We know every PDS gives us a  $C^*$ -PDS with commutative algebra [2]. Following that construction, we are going to use partial actions on upper semicontinuous  $C^*$ -bundles over LCH spaces to construct partial actions on the  $C^*$ -algebras  $C_0(\mathbf{B})$ . The ideals are of the form  $C_0(\mathbf{B}, U) := \{f \in C_0(\mathbf{B}) \mid f(x) = 0_x \text{ if } x \notin U\}$ , for open sets  $U \subset X$ .

**Theorem 2.4.** *Let  $X$  be a LCH space,  $\mathbf{B} = \{B_x\}_{x \in X}$  an upper semicontinuous  $C^*$ -bundle and  $\alpha$  a continuous partial action of  $H$  on  $\mathbf{B}$ . Then  $(C_0(\mathbf{B}), H, \tilde{\alpha})$  is a  $C^*$ -PDS, where*

- (1)  $C_0(\mathbf{B})_t = C_0(\mathbf{B}, X_t)$ , for every  $t \in H$ .
- (2) If  $f \in C_0(\mathbf{B})_{t^{-1}}$  then  $\tilde{\alpha}_t(f)(x) = \alpha_t(f(t^{-1} \cdot x))$  if  $x \in X_t$  and  $0_x$  otherwise.

*Proof.* First of all we have to show that, given  $t \in H$  and  $f \in C_0(\mathbf{B})_{t^{-1}}$ , the function  $\tilde{\alpha}_t(f)$  belongs to  $C_0(\mathbf{B})_t$ . It is clear that  $\tilde{\alpha}_t(f)$  is a section that vanishes outside  $X_t$ . Besides, the function  $X_t \rightarrow \mathbb{R}, x \mapsto \|\tilde{\alpha}_t(f)(x)\|$ , being equal to  $X_t \rightarrow \mathbb{R}, x \mapsto \|f(t^{-1} \cdot x)\|$ , vanishes at infinity.

Clearly  $\hat{\alpha}_t(f)$  is continuous on  $X_t$  and in the interior of the complement of  $X_t$ . To prove the continuity of  $\hat{\alpha}_t(f)$  it suffices to show that given a net  $\{x_i\}_{i \in I} \subset X_t$  converging to a point  $x \notin X_t$ , we have  $\|\tilde{\alpha}_t(f)(x_i)\| \rightarrow 0$ . Notice that the function  $X_{t^{-1}} \rightarrow \mathbb{R}, y \mapsto \|f(y)\|$ , vanishes at infinity and the net  $\{t^{-1} \cdot x_i\}_{i \in I}$  is eventually outside every compact of  $X_{t^{-1}}$ , we conclude  $\|\tilde{\alpha}_t(f)(x_i)\| = \|f(t^{-1} \cdot x_i)\| \rightarrow 0$ .

The next step is to show  $\hat{\alpha}$  is a partial action (Definition 1.1). We omit the proof of this fact because it is an easy task.

To prove  $\{C_0(\mathbf{B})_t\}_{t \in H}$  is a continuous family [5], let  $U$  be an open set of  $C_0(\mathbf{B})$  and fix  $t \in H$  such that  $C_0(\mathbf{B})_t \cap U \neq \emptyset$ . By the Urysohn Lemma we can find  $g \in C_0(\mathbf{B})_t \cap U$  with compact support. As the domain of the partial action on  $X$  is an open set, there is an open set containing  $t$ ,  $V$ , such that  $X_r$  contains the support of  $g$  for every  $r \in V$ . Then  $V$  is an open set containing  $t$  and contained in  $\{r \in H \mid C_0(\mathbf{B})_r \cap U \neq \emptyset\}$ .

Now we deal with the continuity of  $\tilde{\alpha}$ . Let  $\{(t_i, f_i)\}_{i \in I}$  be a net contained in  $\Gamma_{\tilde{\alpha}}$  converging to  $(t, f) \in \Gamma_{\tilde{\alpha}}$ . Given  $\varepsilon > 0$  there exists  $g \in C_c(\mathbf{B})$ , with support contained in  $X_{t^{-1}}$ , such that  $\|f - g\| < \frac{\varepsilon}{3}$  (by the Urysohn Lemma).

We can find an  $i_0 \in I$  such that  $\text{supp}(g) \subset X_{t_i^{-1}}$  and  $\|f_i - g\| < \frac{\varepsilon}{3}$ , for every  $i \geq i_0$ . Then, for every  $i \geq i_0$

$$\begin{aligned} \|\tilde{\alpha}_t(f) - \tilde{\alpha}_{t_i}(f_i)\| &\leq \|\tilde{\alpha}_{t_i}(f_i - g)\| + \|\tilde{\alpha}_{t_i}(g) - \tilde{\alpha}_t(g)\| + \|\tilde{\alpha}_t(f - g)\| \\ &< \frac{2\varepsilon}{3} + \|\tilde{\alpha}_{t_i}(g) - \tilde{\alpha}_t(g)\|. \end{aligned}$$

To complete the proof it suffices to see that  $\lim_i \|\tilde{\alpha}_{t_i}(g) - \tilde{\alpha}_t(g)\| = 0$ . To this purpose let  $D$  be a compact containing  $t \cdot \text{supp}(g)$  on its interior, and contained in  $X_t$ . We may find  $i_1$  (larger than  $i_0$ ) such that  $t_i \cdot \text{supp}(g) \subset D$  and  $D \subset X_{t_i}$ , for every  $i \geq i_1$ . Given  $i \geq i_1$  we have

$$\|\tilde{\alpha}_{t_i}(g) - \tilde{\alpha}_t(g)\| = \sup\{\|\alpha_{t_i}(g(t_i^{-1} \cdot x)) - \alpha_t(g(t^{-1} \cdot x))\| : x \in D\}.$$

As  $D$  is compact, it suffices to prove  $\tilde{\alpha}_{t_i}(g)$  converges point wise to  $\tilde{\alpha}_t(g)$ , which is an easy consequence of Lemma C.18 of [13] and the continuity of  $\alpha$  and  $g$ .  $\square$

The definitions of proper, free and commuting partial actions on bundles are the following ones.

**Definition 2.5.** Given an upper semicontinuous  $C^*$ -bundle over a LCH space and a partial action of a LCH group on the bundle, we say the partial action is *free*, *proper*, *has closed graph* or *has closed domain* if the partial action on the base space has the respective property. Similarly, given two partial actions on an upper semicontinuous  $C^*$ -bundle we say they commute if the partial actions on the total and base space commute.

Relating the concepts of enveloping action, in the contexts of  $C^*$ -algebras and bundles, we have the following result.

**Theorem 2.6.** *Let  $\alpha$  be a partial action of  $H$  on the upper semicontinuous  $C^*$ -bundle  $\mathbf{B} = \{B_x\}_{x \in X}$ . If  $X$  is LCH,  $\alpha$  has closed graph,  $\alpha^e$  is the enveloping action of  $\alpha$  and  $\mathbf{B}^e$  the enveloping bundle, then  $(C_0(\mathbf{B}^e), H, \tilde{\alpha}^e)$  is the enveloping system of  $(C_0(\mathbf{B}), H, \tilde{\alpha})$  (Definition 2.3 of [2]). So  $C_0(\mathbf{B}) \rtimes_{\tilde{\alpha}} H$  is a hereditary and full sub  $C^*$ -algebra of  $C_0(\mathbf{B}^e) \rtimes_{\tilde{\alpha}^e} H$ . In particular those crossed products are strongly Morita equivalent.*

*Proof.* By Theorem 2.3 we may suppose  $X \subset X^e$ ,  $\mathbf{B} \subset \mathbf{B}^e$ ,  $p^e(\mathbf{B}) = X$  and that  $p$  is the restriction of  $p^e$  to  $\mathbf{B}$ . Now, by Lemma 1.8,  $X^e$  is LCH. This considerations allows us to identify the bundle  $\mathbf{B}$  with the restriction of  $\mathbf{B}^e$  to  $X$ , which gives  $C_0(\mathbf{B}) = C_0(\mathbf{B}^e, X)$ . We have identified  $C_0(\mathbf{B})$  with an ideal of  $C_0(\mathbf{B}^e)$ .

We also have, for every  $t \in H$ ,

$$\begin{aligned} \tilde{\alpha}_t^e(C_0(\mathbf{B}^e, X)) \cap C_0(\mathbf{B}^e, X) &= C_0(\mathbf{B}^e, t \cdot X) \cap C_0(\mathbf{B}^e, X) = C_0(\mathbf{B}^e, X \cap t \cdot X) \\ &= C_0(\mathbf{B}^e, X_t) = C_0(\mathbf{B})_t; \end{aligned}$$

and clearly the restriction of  $\tilde{\alpha}_t^e$  to  $C_0(\mathbf{B})_{t^{-1}}$  equals  $\tilde{\alpha}_t$ . So, using Corollary 1.3 of [3], the only thing that remains to be showed is that the space generated by the  $\tilde{\alpha}^e$ -orbit of  $C_0(\mathbf{B})$  is dense in  $C_0(\mathbf{B}^e)$ .

We show every continuous section with compact support of  $\mathbf{B}^e$  is a finite sum of points in the orbit of  $C_0(\mathbf{B})$ . Fix  $f \in C_c(\mathbf{B}^e)$ . The support of  $f$  has an open cover by sets of the form  $t \cdot X$ ,  $t$  varying in  $H$ . We can find  $t_1, \dots, t_n \in H$  and  $h_1, \dots, h_n \in C_c(X^e)$  such that:  $0 \leq h_1 + \dots + h_n \leq 1$ , the support of  $h_i$  is contained



in  $t_i \cdot X$  ( $i = 1, \dots, n$ ) and  $h_1(x) + \dots + h_n(x) = 1$  if  $x \in \text{supp}(f)$ . Defining  $f_i(x) = h_i(x)f(x)$  ( $i = 1, \dots, n$ ), we have  $f = f_1 + \dots + f_n$  and  $g_i := \widetilde{\alpha}_{t_i^{-1}}^e(f_i) \in C_c(\mathbf{B})$  for every  $i = 1, \dots, n$ . Besides,  $f = \widetilde{\alpha}_{t_1}^e(g_1) + \dots + \widetilde{\alpha}_{t_n}^e(g_n)$ , that gives the desired result.  $\square$

We can reproduce most of the results of Section 1 in this context. For example, the next Theorem is a direct consequence of Lemma 1.2.

**Theorem 2.7.** *Given an upper semicontinuous  $C^*$ -bundle  $\mathbf{B}$  and commutative partial actions,  $(\alpha, \cdot)$  and  $(\beta, \star)$  of  $H$  and  $K$  on  $\mathbf{B}$ , respectively, the pair  $(\alpha, \cdot) \times (\beta, \star) := (\alpha \times \beta, \cdot \times \star)$  is a partial action of  $H \times K$  on  $\mathbf{B}$ .*

Writing  $\alpha = (\alpha, \cdot)$  and  $\beta = (\beta, \star)$ , the product  $\alpha \times \beta$  is the one defined in the previous Theorem.

**2.1. Orbit bundle.** There is a notion of “orbit bundle”, analogous to the notion of “orbit space”, but to construct it we have to consider proper and free partial actions.

Fix an upper semicontinuous  $C^*$ -bundle over a LCH space,  $\mathbf{B} = \{B_x\}_{x \in X}$ , and a proper and free partial action,  $\alpha$ , of  $H$  on  $\mathbf{B}$ . Let  $B/H$  and  $X/H$  be the orbit spaces and  $\pi_B : B \rightarrow B/H$  and  $\pi_X : X \rightarrow X/H$  be the orbit maps. As  $p$  is a morphism of partial actions, there is a unique continuous (also open and surjective) function  $p_\alpha : B/H \rightarrow X/H$  such that  $\pi_X \circ p = p_\alpha \circ \pi_B$ .

We want to equip  $B/H$  with operations making  $\mathbf{B}/H := (B/H, X/H, p_\alpha)$  an upper semicontinuous  $C^*$ -bundle. To do this first notice that, given  $Hx \in X/H$ , the fiber  $(B/H)_{xH}$  is homeomorphic to  $B_x$  through the restriction of  $\pi_B$  to  $B_x$  (because the partial action on  $X$  is free). Call that map  $h_x : B_x \rightarrow (B/H)_{xH}$ . Define the structure of  $C^*$ -algebra of  $(B/H)_{xH}$  in such a way that  $h_x$  is an isomorphism of  $C^*$ -algebras. This definition is independent of the choice of  $x$  because, if  $Hy = Hx$ , then  $h_y^{-1} \circ h_x : B_x \rightarrow B_y$  is an isomorphism. As it is the restriction of  $\alpha_t$  to  $B_x$ ,  $t \in H$  being the unique such that  $x \in X_{t^{-1}}$  and  $t \cdot x = y$ .

The norm of  $B/H$  is the function  $\| \cdot \| : B/H \rightarrow \mathbb{R}$ ,  $Ha \mapsto \|a\|$ . To prove it is upper semicontinuous let  $\varepsilon$  be a positive number. The set  $\{Hb \in B/H \mid \|Hb\| < \varepsilon\}$  is open because it equals the open set  $\pi_B(\{b \in B \mid \|b\| < \varepsilon\})$ . Similarly, we prove  $\| \cdot \| : B/H \rightarrow \mathbb{R}$  is continuous if  $\| \cdot \| : B \rightarrow \mathbb{R}$  is continuous.

To prove the continuity of the product and the sum name  $D$  the set of points  $(a, b) \in B \times B$  such that  $Ha = Hb$ . If  $(a, b) \in D$  there is a unique  $t \in H$ , which we name  $t(p(a), p(b))$ , such that  $p(a) \in X_{t^{-1}}$  and  $t \cdot p(a) = p(b)$ . Hence,  $\alpha_t(a)$  and  $b$  are in the same fiber, we define  $S(a, b) := \alpha_t(a) + b$  and  $P(a, b) := \alpha_t(a)b$ .

To prove the continuity of  $S$  and  $P$  we only have to prove the continuity of the function

$$F : \{(x, y) \in X \times X : Hx = Hy\} \rightarrow \Gamma(X, \alpha), \quad F(x, y) = (t(x, y), x).$$

Call  $D_X$  the domain of  $F$ .

Consider the function  $R : \Gamma(X, \alpha) \rightarrow X \times X$  given by  $R(t, x) = (x, t \cdot x)$ , this is a continuous, proper and injective function between LCH spaces. Such functions are homeomorphisms over its image, but the image of  $R$  is  $D_X$  and  $F = R^{-1}$ . So,  $F$  is continuous.

Once we have proved the continuity of  $S$  and  $P$ , using the freeness of the partial action on  $X$ , we prove they are constant in the classes of the equivalence relation

on  $D : (a, b) \sim (c, d)$  if  $Ha = Hc$  and  $Hb = Hd$ . The space  $D/\sim$  is (homeomorphic to)  $D' := \{(a, b) \in B/H \times B/H : p_\alpha(a) = p_\alpha(b)\}$ , and the functions defined by  $S$  and  $P$  on  $D'$  are exactly the sum and product of  $B/H$ . We have proved they are continuous, for the rest of the operations we proceed in a similar way.

The last step, to show  $\mathbf{B}/H$  is an upper semicontinuous  $C^*$ -bundle is to prove it satisfies the following property: for every net  $\{b_i\}_{i \in I} \subset B/H$  such that  $\|b_i\| \rightarrow 0$  and  $p_\alpha(b_i) \rightarrow z$ , for some  $z \in X/H$ , we have  $b_i \rightarrow 0_z$ .

Let  $\{b_i\}_{i \in I}$  be a net as before, it suffices to show it has a subnet converging to  $0_z$ . There is a net in  $B$ ,  $\{a_i\}_{i \in I}$ , such that  $b_i = Ha_i$  for every  $i \in I$ . We have that  $Hp(a_i) = p_\alpha(b_i) \rightarrow Hx$ , where  $x \in X$  is such that  $Hx = z$ . As the orbit map  $X \rightarrow X/H$  is open and surjective, Proposition 13.2 Chapter II of [7] implies there is a subnet  $\{a_{i_j}\}_{j \in J}$  and a net  $\{t_j\}_{j \in J} \subset H$  such that  $p(a_{i_j}) \in X_{t_j}^{-1}$  and  $t_j \cdot p(a_{i_j}) \rightarrow x$ . This implies  $a_{i_j} \in {}_{t_j}^{-1}B$  and  $p(\alpha_{t_{i_j}}(a_{i_j})) \rightarrow x$ . But also  $\|\alpha_{t_{i_j}}(a_{i_j})\| = \|b_{i_j}\| \rightarrow 0$ , so,  $\alpha_{t_{i_j}}(a_{i_j}) \rightarrow 0_x$ . Finally, as  $b_{i_j} = H\alpha_{t_{i_j}}(a_{i_j})$ ,  $\pi_B(0_x) = H0_x = 0_z$  and  $\pi_B$  is continuous,  $0_z$  is a limit point of  $\{b_{i_j}\}_{j \in J}$ .

**Definition 2.8.** The *orbit bundle* of  $\mathbf{B}$  by  $\alpha$  is the upper semicontinuous  $C^*$ -bundle  $\mathbf{B}/H$  constructed before.

**Example 2.9.** Consider the situation of Example 2.2 where the action of  $H$  on  $A$  is the trivial one ( $\gamma_t = \text{id}_A$  for every  $t \in H$ ) and the system  $(X, H, \cdot)$  is free and proper. Then the quotient bundle  $\mathbf{B}/H$  is isomorphic to the trivial bundle  $A \times X/H$ . Notice that  $C_0(\mathbf{B}/H)$  is isomorphic to  $C_0(X/H, A)$ , the set of continuous functions from  $X/H$  to  $A$  vanishing at infinity.

Our next goal is to identify  $C_0(\mathbf{B}/H)$  with a  $C^*$ -sub algebra of  $C_b(\mathbf{B})$ . Every function  $f \in C_b(\mathbf{B})$ , which is also a morphism of partial actions, induces a continuous and bounded section  $\text{Ind}_b(f) : X/H \rightarrow B/H$ , given by  $Hx \mapsto Hf(x)$ .

The induced algebra  $\text{Ind}_b(\mathbf{B}, \alpha)$  is the subset of  $C_b(\mathbf{B})$  formed by all the sections which are also morphism of partial actions. There is a natural map

$$\text{Ind}_b : \text{Ind}_b(\mathbf{B}, \alpha) \rightarrow C_b(\mathbf{B}/H), \quad f \mapsto \text{Ind}_b(f).$$

Similarly, the algebra  $\text{Ind}_0(\mathbf{B}, \alpha)$  is the pre image of  $C_0(\mathbf{B}/H)$  under  $\text{Ind}_b$ . The function  $\text{Ind}_0$  is simply the restriction of  $\text{Ind}_b$  to  $\text{Ind}_0(\mathbf{B}, \alpha)$ .

In fact, the induced algebras are  $C^*$ -sub algebras of  $C_b(\mathbf{B})$ . To prove this it suffices to show  $\text{Ind}_b(\mathbf{B}, \alpha)$  is a  $C^*$ -sub algebra and to notice  $\text{Ind}_b$  is a morphism of  $C^*$ -algebras.

The non trivial fact is that  $\text{Ind}_b(\mathbf{B}, \alpha)$  is closed in  $C_b(\mathbf{B})$ . Assume  $\{f_n\}_{n \in \mathbb{N}}$  is a sequence contained in  $\text{Ind}_b(\mathbf{B}, \alpha)$  converging to  $f$ . Choose some  $t \in H$  and  $x \in X_{t^{-1}}$ . Even if  $B$  is not Hausdorff,  $B_x$  and  $B_{t \cdot x}$  are, so we have the following equalities

$$\alpha_t(f(x)) = \lim_n \alpha_t(f_n(x)) = \lim_n f_n(t \cdot x) = f(t \cdot x).$$

**Theorem 2.10.** *The functions*

$$\text{Ind}_b : \text{Ind}_b(\mathbf{B}, \alpha) \rightarrow C_b(\mathbf{B}/H) \text{ and } \text{Ind}_0 : \text{Ind}_0(\mathbf{B}, \alpha) \rightarrow C_0(\mathbf{B}/H)$$

*are isomorphism of  $C^*$ -algebras.*

*Proof.* The only thing to prove is that  $\text{Ind}_b$  is surjective (it is injective because is an isometry). Fix  $g \in C_b(\mathbf{B}/H)$ , we will construct  $f \in \text{Ind}_b(\mathbf{B}, \alpha)$  such that  $\text{Ind}_b(f) = g$ .

As the action on the base space is free, for every  $x \in X$  there is a unique  $f(x) \in B_x$  such that  $Hf(x) = g(Hx)$ . Clearly  $f$  is a bounded section.

To prove  $f$  is continuous, let  $\{x_i\}_{i \in I}$  be a net contained in  $X$  converging to  $x \in X$ . It suffices to find a subnet  $\{x_{i_j}\}_{j \in J}$  such that  $f(x_{i_j}) \rightarrow f(x)$ . By the continuity of  $g$  the net  $\{Hf(x_i)\}_{i \in I}$  has  $Hf(x)$  as a limit point. As the orbit map  $B \rightarrow B/H$  is open, there is a subnet  $\{x_{i_j}\}_{j \in J}$  and a net  $\{t_j\}_{j \in J}$  such that  $\{(t_j, f(x_{i_j}))\}_{j \in J} \subset \Gamma(B, \alpha)$  and  $\alpha_{t_j}(f(x_{i_j})) \rightarrow f(x)$ . This implies  $\{(t_j, x_{i_j})\}_{j \in J} \subset \Gamma(X, \alpha)$  and  $t_j \cdot x_{i_j} \rightarrow x$ . Then  $t_j = t(x_{i_j}, t_j \cdot x_{i_j}) \rightarrow t(x, x) = e$  (see the construction of the orbit bundle in Section 2.1). Finally, the net  $\{f(x_{i_j})\}_{j \in J}$ , being equal to  $\{\alpha_{t_j}^{-1} \alpha_{t_j}(f(x_{i_j}))\}_{j \in J}$ , has  $f(x)$  as a limit point.

It remains to prove  $f$  is a morphism of partial actions. Clearly  $f(X_t) \subset {}_t B$  for every  $t \in H$ . Now take  $t \in H$  and  $x \in X_{t^{-1}}$ . The points  $f(t \cdot x)$  and  $\alpha_t(f(x))$  are, both, the unique point of  $B_{t \cdot x}$  in the class of  $g(Hx)$ , so they are equal.  $\square$

**Theorem 2.11.** *Let  $\mathbf{B} = \{B_x\}_{x \in X}$  be an upper semicontinuous  $C^*$ -bundle over a LCH space and  $\alpha$  and  $\beta$  be partial actions of  $H$  and  $K$  on  $\mathbf{B}$ , respectively. If  $\alpha$  is free and proper, then  $(\text{Ind}_0(\mathbf{B}, \alpha), K, \hat{\beta})$  is a  $C^*$ -PDS where*

- (1)  $\text{Ind}_0(\mathbf{B}, \alpha)_t := \{f \in \text{Ind}_0(\mathbf{B}, \alpha) : x \mapsto \|f(x)\| \text{ vanishes outside } X_t^H\}$ .
- (2) *For every  $f \in \text{Ind}_0(\mathbf{B}, \alpha)_{t^{-1}}$   $\hat{\beta}_t(f)(x) = \beta_t(f(t^{-1} \star x))$  if  $x \in X_t^K$  and 0 otherwise.*

*Proof.* Let  $\mathbf{B}/H$  be the orbit bundle. As  $\alpha$  commutes with  $\beta$ , using Lemmas 1.3 and 1.11, we define a partial action,  $\mu$ , of  $K$  on  $\mathbf{B}/H$ .

By Theorem 2.4,  $\mu$  defines a  $C^*$ -PDS  $(C_0(\mathbf{B}/H), K, \tilde{\mu})$ . Lemma 2.10 ensures the map  $\text{Ind}_0 : \text{Ind}_0(\mathbf{B}, \alpha) \rightarrow C_0(\mathbf{B}/H)$  is an isomorphism. Notice  $\text{Ind}_0(\mathbf{B}, \alpha)_t$  is the pre image of  $C_0(\mathbf{B}/H)_t$ . The partial action of the thesis is the unique making  $\text{Ind}_0 : \hat{\beta} \rightarrow \tilde{\mu}$  an isomorphism of partial actions.  $\square$

### 3. MORITA EQUIVALENCE

In our last section we prove our main theorem, which is a generalization of Raeburn's and Green's Symmetric Imprimitivity Theorems [10, 11]. The first task is to translate Raeburn's result to the language of actions on bundles.

Consider two  $C^*$ -DS  $(A, H, \gamma)$  and  $(A, K, \delta)$ , and two proper and free DS  $(X, H, \cdot)$  and  $(X, K, \star)$ . Assume also that the actions on  $A$  and  $X$  commute. On the trivial bundle  $\mathbf{B} = A \times X$  define the actions of  $H$  and  $K$  as in Example 2.2, call them  $\alpha$  and  $\beta$ , respectively.

Let  $\text{Ind } \gamma$  be the induced  $C^*$ -algebra defined as in [10]. We have an isomorphism  $\rho : \text{Ind } \gamma \rightarrow \text{Ind}_0(\mathbf{B}, \alpha)$ , given by  $\rho(f)(x) = (f(x), x)$ . This isomorphism takes the action of  $K$  on  $\text{Ind } \gamma$  (as defined on [10]) into the action  $\hat{\beta}$ . By using Raeburn's Theorem we conclude that  $\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\hat{\beta}} K$  is strongly Morita equivalent to  $\text{Ind}_0(\mathbf{B}, \beta) \rtimes_{\hat{\alpha}} H$ . Our purpose is to give a version of this result for partial actions. We will write  $A \sim_M B$  whenever  $A$  and  $B$  are strongly Morita equivalent  $C^*$ -algebras [12].

**3.1. The main Theorem.** From now on we work with two LCH topological groups,  $H$  and  $K$ , an upper semicontinuous  $C^*$ -bundle with LCH base space,  $\mathbf{B} = \{B_x\}_{x \in X}$ , and two continuous, free, proper and commuting partial actions,  $\alpha$  and  $\beta$ , of  $H$  and  $K$  on  $\mathbf{B}$ , respectively.

We want to give conditions under which we can say that  $\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\widehat{\beta}} K$  is strongly Morita equivalent to  $\text{Ind}_0(\mathbf{B}, \beta) \rtimes_{\widehat{\alpha}} H$ . For global actions, with some additional hypotheses on the group and the base space, this is proved in [4], [9] or [8]. In fact, the proof of the next Theorem is a minor modification of Raeburn's proof the Symmetric Imprimitivity Theorem [10].

**Theorem 3.1.** *If  $\alpha$  and  $\beta$  are global actions then*

$$\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\widehat{\beta}} K \sim_M \text{Ind}_0(\mathbf{B}, \beta) \rtimes_{\widehat{\alpha}} H.$$

*Proof.* Define  $E := C_c(H, \text{Ind}_0(\mathbf{B}, \beta))$  and  $F := C_c(K, \text{Ind}_0(\mathbf{B}, \alpha))$ , viewed as dense  $*$ -sub-algebras of the respective crossed products. Define also  $Z := C_c(\mathbf{B})$ , which will be an  $E - F$ -bimodule with inner products; whose completion implements the equivalence between  $\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\widehat{\beta}} K$  and  $\text{Ind}_0(\mathbf{B}, \beta) \rtimes_{\widehat{\alpha}} H$ .

For  $f, g \in Z$ ,  $b \in E$  and  $c \in F$  define

$$(3.1) \quad b \cdot f(x) := \int_H b(s)(x) \widetilde{\alpha}_s(f)(x) \Delta_H(s)^{1/2} ds,$$

$$(3.2) \quad f \cdot c(x) := \int_K \widetilde{\beta}_t(fc(t^{-1}))(x) \Delta_K(t)^{-1/2} dt,$$

$$(3.3) \quad {}_E\langle f, g \rangle(s)(x) := \Delta_H(s)^{-1/2} \int_K \widetilde{\beta}_t(f \widetilde{\alpha}_s(g^*)) (x) dt,$$

$$(3.4) \quad \langle f, g \rangle_F(t)(x) := \Delta_K(t)^{-1/2} \int_H \widetilde{\alpha}_s(f^* \widetilde{\beta}_t(g)) (x) ds.$$

The integration is with respect to left invariant Haar measures;  $\Delta_H$  and  $\Delta_K$  are the modular functions of the groups. Here  $\widetilde{\alpha}$  and  $\widetilde{\beta}$  are the partial actions defined on Theorem 2.4.

We now justify the fact that  $b \cdot f \in Z$ . The function  $H \rightarrow C_0(\mathbf{B})$ , given by  $s \mapsto b(s) \widetilde{\alpha}_s(f) \Delta_H(s)^{1/2}$ , is continuous (Theorem 2.4). Besides, its support is contained in the support of  $b$  and so we can integrate it. This integral is exactly  $b \cdot f$ . Finally, notice  $\text{supp}(b \cdot f) \subset \{s \cdot x : (s, x) \in \text{supp}(b) \times \text{supp}(f)\}$ , the last being a compact set.

To prove (3.3) defines an element of  $E$  we proceed as follows. Fixed  $s \in H$  and  $x \in X$  the function  $K \rightarrow B_x$ , given by  $t \mapsto \widetilde{\beta}_t(f \widetilde{\alpha}_s(g^*)) (x)$ , is continuous with support contained in the compact  $\{t \in K : t^{-1} \star x \in \text{supp}(f)\}$ . So, the function is integrable. The value of that integral is  ${}_E\langle f, g \rangle(s)(x)$ .

We now prove  ${}_E\langle f, g \rangle$  is continuous, what we do locally. Fix some  $s_0 \in H$  and  $x_0 \in X$ . Take compact neighbourhoods,  $V$  of  $s_0$  and  $W$  of  $x_0$ . The bundle  $\mathbf{B}_W$  will be the restriction of  $\mathbf{B}$  to  $W$ . Define the function  $F : V \times H \rightarrow C(\mathbf{B}_W)$  by  $F(s, t)(x) = \widetilde{\beta}_t(f \widetilde{\alpha}_s(g^*)) (x)$ . As the action of  $K$  on  $X$  is proper,  $F$  has compact support. By integrating, with respect to the second coordinate, we get a continuous function  $R \in C(V, C(\mathbf{B}_W))$ , defined by  $R(s) = \int_K F(s, t) d\mu_K(t)$  ([7] II.15.19).

Fixing  $(s, x) \in V \times W$ , we have  $R(s)(x) = {}_E\langle f, g \rangle(s)(x)$ . From this follows the continuity of  ${}_E\langle f, g \rangle$ .

An easy calculation shows  ${}_E\langle f, g \rangle(s)(t \cdot x) = \beta_t({}_E\langle f, g \rangle(s)(x))$ , for every  $t \in K$  and  $x \in X$ . Besides, if  ${}_E\langle f, g \rangle(s)(x) \neq 0_x$ , then  $x$  belongs to the  $K$ -orbit of  $\text{supp}(f)$ , and  $s$  to the compact set  $\{s \in H : s \cdot \text{supp}(g) \cap \text{supp}(f) \neq \emptyset\}$ . We have proved that  ${}_E\langle f, g \rangle \in E$ .

The computations needed to prove equations (3.1)-(3.4) define an equivalence bi-module are the same as in [10] or [13]. For the construction of the approximate

unit, analogous to that of Lemma 1.2 of [10], follow the proof of Proposition 4.5 of [13], recalling  $\text{Ind}_H^P(A, \beta)$  plays the role of our  $\text{Ind}_0(\mathbf{B}, \beta)$ .  $\square$

Our next step is to let  $\alpha$  and  $\beta$  to be partial, but to have the same result we need additional hypotheses, which are trivially satisfied in the previous case.

Let  $\alpha \times \beta$  be the partial action given by Theorem 2.7. Now, by Theorem 2.3, we have an enveloping action  $(\alpha \times \beta)^e$  and an enveloping bundle  $\mathbf{B}^e$ . We can assume  $\mathbf{B}$  is the restriction of  $\mathbf{B}^e$  to  $X \subset X^e$ .

For the action given by  $(\alpha \times \beta)^e$  on  $X^e$  we will use the notation  $(s, t)x$ , for  $(s, t) \in H \times K$  and  $x \in X^e$ .

Define  $\sigma$  and  $\tau$  as the restriction of  $(\alpha \times \beta)^e$  to  $H$  and  $K$ , respectively (identify  $H$  with  $H \times \{e\} \subset H \times K$ ). It is immediate that  $(\alpha \times \beta)^e = \sigma \times \tau$ ,  $\sigma$  and  $\tau$  commute, and that  $\alpha$  ( $\beta$ ) is the restriction of  $\sigma$  ( $\tau$ ) to  $\mathbf{B}$ .

The next is the main Theorem of this article.

**Theorem 3.2.** *If  $\alpha \times \beta$  has closed graph and  $\sigma$  and  $\tau$  are proper then*

$$\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\hat{\beta}} K \sim_M \text{Ind}_0(\mathbf{B}, \beta) \rtimes_{\hat{\alpha}} H.$$

*Proof.* To show that  $\sigma$  (and also  $\tau$ ) is free. Assume  $(s, e)x = x$  for some  $s \in H$  and  $x \in X^e$ . As  $X^e$  is the  $H \times K$ -orbit of  $X$ , there exists  $(h, k) \in H \times K$  such that  $(h, k)x \in X$ . Notice that  $(hsh^{-1}, e)(h, k)x = (h, k)x \in X \cap (hsh^{-1}, e)^{-1}X$ , so,  $hsh^{-1} \cdot (h, k)x = (h, k)x$  and  $hsh^{-1} = e$ . We conclude  $s = e$ .

As  $\alpha \times \beta$  has closed graph,  $\mathbf{B}^e$  is an upper semicontinuous  $C^*$ -bundle over a LCH space. The hypotheses, together with Theorems 2.11 and 3.1, imply  $\text{Ind}_0(\mathbf{B}^e, \sigma) \rtimes_{\hat{\tau}} K$  is strongly Morita equivalent to  $\text{Ind}_0(\mathbf{B}^e, \tau) \rtimes_{\hat{\sigma}} H$ . The proof of our Theorem will be completed if we can show that  $\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\hat{\beta}} K$  is strongly Morita equivalent to  $\text{Ind}_0(\mathbf{B}^e, \sigma) \rtimes_{\hat{\tau}} K$ , because, by symmetry, the same will hold changing  $\alpha$  for  $\beta$ ,  $\sigma$  for  $\tau$  and  $H$  for  $K$ .

Tracking back the construction of  $\hat{\beta}$  and  $\hat{\tau}$ , to Theorem 2.11, we notice that  $\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\hat{\beta}} K$  is isomorphic to  $C_0(\mathbf{B}/H) \rtimes_{\hat{\mu}} K$  and  $\text{Ind}_0(\mathbf{B}^e, \sigma) \rtimes_{\hat{\tau}} K$  is isomorphic to  $C_0(\mathbf{B}^e/H) \rtimes_{\hat{\nu}} K$ . Here  $\mu$  and  $\nu$  are the partial actions of  $K$  on  $\mathbf{B}/H$  and  $\mathbf{B}^e/H$  given by Lemma 1.3, respectively. Meanwhile,  $\tilde{\mu}$  and  $\tilde{\nu}$  are the one given by Theorem 2.4. Putting all together, by Theorem 2.6, it suffices to prove  $\nu$  is the enveloping action of  $\mu$ .

Consider the map  $B \rightarrow B^e/H$ , given by  $b \mapsto Hb$ . This is an open and continuous map, it is also constant in the  $\alpha$ -orbits. So it defines a unique map  $F : B/H \rightarrow B^e/H$ , given by  $Hb \mapsto Hb$  (this is not the identity map). It turns out this function is continuous, open, injective and maps fibers into fibers. In an analogous way we define  $f : X/H \rightarrow X^e/H$ , which has the same topological properties.

Recalling the construction of  $\mu$  and  $\nu$ , it is easy to show  $(F, f) : \mu \rightarrow \nu$  is a morphism. To show that  $\mu^e = \nu$  it suffices to prove only two things. Namely, that  $f((X/H)_t) = f(X/H) \cap tf(X/H)$  for every  $t \in K$  (we adopted the notation  $tz$  for the action of  $t \in K$  on  $z \in X^e/H$ ) and that the  $K$ -orbit of  $f(X/H)$  is  $X^e/H$ .

For the first one notice that

$$f((X/H)_t) = HX_t^K = HX \cap H(e, t)X = HX \cap tHX = f(X/H) \cap tf(X/H).$$

The second equality of the previous formula is not immediate, but the inclusion  $\subset$  is. For the other one assume  $y \in HX \cap H(e, t)X$ . Then there exists  $x, z \in X$  such that  $y = Hx = H(e, t)z$ . There is some  $s \in H$  such that  $x = (s, e)(e, t)z = (s, t)z$ . So  $x \in X \cap (s, t)X = s \cdot (X_s^H \cap X_t^K)$ ,  $(s^{-1}, e)x \in X_t^K$  and  $y = K(s^{-1}, e)x \in HX_t^K$ .

To show  $X^e/H$  is the  $K$ -orbit of  $f(X/H)$ , notice that

$$\bigcup_{t \in K} tf(X/H) = \bigcup_{t \in K} tHX = H \bigcup_{s \in H} \bigcup_{t \in K} (s, t)X = HX^e = X^e/H.$$

We have proved  $\nu$  is the enveloping action of  $\mu$ , by Theorem 2.6  $C_0(\mathbf{B}/H) \rtimes_{\tilde{\mu}} K$  is strongly Morita equivalent to  $C_0(\mathbf{B}^e/H) \rtimes_{\tilde{\nu}} K$ . This completes the proof of our main theorem.  $\square$

The next Theorem is a consequence of the previous one, it has the advantage of not making any mention to  $\sigma$  nor  $\tau$ .

**Theorem 3.3.** *If  $\alpha$  and  $\beta$  have closed domain then*

$$\text{Ind}_0(\mathbf{B}, \alpha) \rtimes_{\hat{\beta}} K \sim_M \text{Ind}_0(\mathbf{B}, \beta) \rtimes_{\hat{\alpha}} H.$$

*Proof.* We check the hypotheses of the previous theorem are satisfied. To show  $\alpha \times \beta$  has closed graph notice it has closed domain (Lemma 1.6) and use Lemma 1.5. Finally, we only have to show  $\sigma$  and  $\tau$  are proper. It is enough to show  $\sigma$  is proper, for that purpose we use Lemma 1.9.

Let  $\{(s_i, x_i)\}$  be a net in  $H \times X^e$  such that  $\{(x_i, (s_i, e)x_i)\}$  converges to the point  $(x, y) \in X^e \times X^e$ . It is enough to show  $\{s_i\}_{i \in I}$  has a converging subnet. We may assume  $(s, t)x \in X$  and  $(h, k)y \in X$ , for some  $(h, k), (s, t) \in H \times K$ .

There is an  $i_0$  such that, for  $i \geq i_0$ ,  $(s, t)x_i$  and  $(h, k)(s_i, e)x_i$  belong to  $X$ . For  $i \geq i_0$  define  $u_i = (s, t)x_i$ . By the construction of  $\alpha \times \beta$  and because  $(hs_i s^{-1}, kt^{-1})u_i \in X$ , we have that  $u_i$  is an element of the clopen set  $tk^{-1} \star (X_{k^{-1}t}^K \cap X_{ss_i h^{-1}}^H)$ . Defining  $v_i := (e, k^{-1}t)u_i$  for every  $i \geq i_0$ , we have that  $v_i \in X$ . So, the limit  $\lim_i v_i$  is an element of  $X$  (recall  $X$  is clopen in  $X^e$ ).

The net  $\{(hs_i s^{-1}, v_i)\}_i$  is contained in  $\Gamma(X, \alpha)$  and  $\{(v_i, hs_i s^{-1} \cdot v_i)\}_i$  has a limit point. Then  $\{hs_i s^{-1}\}_i$  has a converging subnet, and so  $\{s_i\}_i$  has a converging subnet. We conclude  $\sigma$  is proper, and we are done.  $\square$

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